Chapter 1

Preliminaries

In this chapter we will describe give some background material that is needed for understanding the developments in Chapter 2 and beyond. Readers with some experience with interacting particle sysems can probably skip this chapter and refer back to it as needed.

Interacting particle systems are a type of Markov process, so Section 1.1.gives a quick introduction to semigroups and generators for continuous time processes on discrete state spaces. In Liggett's (1985) book this approach is used to construct interacting particle systems but here we build our systems them from collections of Poisson processes. In Section 1.2 we give a simple construction due to Harris (1972) that can be used to construct any translation invariant process with finite range. In Section 1.3 we describe **a graphical representation**, which is in essence an oriented percolation process on $\mathbb{Z}^d \times (-\infty, \infty)$. Set valued processes constructed in this way are **additive**,

$$\xi_t^{A\cup B} = \xi_t^A \cup |xi_t^B$$

In words, the set of occupied sites starting from $A \cup B$ occupied is the union of the occupied sites starting from A and B. Additive processes always have a dual process ζ_t that satisfies

$$P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \zeta_t^B)$$

Section 1.3 describes some of the most important examples can be constructed from a graphaical representation: the contact process, the voter model, and variants with nonlinear birth and/or death rates, though as the reader will see there are severe restrictions on the nonlinear rates that can be constructed. For example, it is impossible to construct "sexual reproduction," in which the birth rate at a site is 0 unless at least two neighbors are occupied, is impossible.

In Section 1.4 we introduce **attractive processes**, which are monotone in the sense that if $A \subset B$ then the two processes can be constructed on the same space so that $\xi_t^A \subset \xi_t^B$ for all $t \geq 0$. These processes have the useful property that if we let ξ_t^1 be the system starting from all sites occupied then

$$\lim_{t\to\infty} \xi_t^1 \equiv \xi_\infty^1$$

exists and is a stationary distribution. Similarly if we let $\xi_t^\emptyset$ be the process starting from $\emptyset$ then

$$\lim_{t\to\infty} \xi_t^\emptyset \equiv \xi_\infty^\emptyset$$

exists and is a stationary distribution. If $\xi_\infty^1 = \xi_\infty^\emptyset$ the stationary distribution is unique. This is not as exciting as it may seem, since in some examples $\emptyset$ is an absorbing state, so $\xi_\infty^\emptyset = \emptyset$ with probability 1, and interest focuses on finding nontrivial stationary distributions

In Section 1.5 we introduce **oriented percolation on**

$$\mathcal{L} = \{(x,n) \in \mathbb{Z}^2 : x+n \text{ is even } \}$$

with edges connecting $(x,n) \to (x+1,n+1)$ and $(x,n) \to (x-1,n+1)$. Here the first coordinate is space and the second is time. We use the **contour method** to show that if the probability an edge is open is $> 8/9$ then there is probability of oriented bond percolation on $\mathcal{L}$.

Given an initial set of occupied sites $A \subset 2\mathbb{Z}$ we let

$$\xi_n^A = \{(y,n) : (x,0) \to (y,n) \text{ for some } x \in A\}$$

ξ_n^A is a **discrete time contact process**. If we think of ξ_n^A as the locations occupied by plants in year n, then no plant survives from one year to the next. A plant at x in year n independently produces offspring at $x+1$ and $x-1$ with probability p in year $n+1$. All sites in year $n+1$ that receive at least one offspring are deemed occupied, the others are vacant. Reinterpreting the result for percolation we can say that the discrete time contact process survives with positive probability starting ftrom $A = \{0\}$ if $p > 8/9$.

One of our main techniques in the book is the **block construction**, in which the existence of phase transitions is proved by comparing interacting particle systems with oriented percolation.

1.1 A few words about Markov processes

c:Markovint

In this section, we will mostly state definitions to establish terminology. If you need more explanation consult Section 1.1 of Liggett (1985). In most cases our processes will have state space F^S where F is a finite set of states and S is a countable set of sites, usually $\mathbb{Z}^d$. In this brief introduction to the theory of Markov processes we will restrict our attention to $F = \{0, 1\}$. The state space, F^S, which Liggett (1985) denotes by $\mathbf{X}$, is endowed with the product topology, which makes $\mathbf{X}$ compact. A Markov process on $\mathbf{X}$ is a collection $\{P^\eta, \eta \in \mathbf{X}\}$ of probability measures on the Skorokhod space $D[0, \infty)$ of cádlág paths, which is French for right continuous with left limits from $[0, \infty) \to \mathbf{X}$. To formulate the **Markov property** we let $\mathcal{F}_s$ be the σ-field generated by the process up to time s

$$P^\eta(\eta_{s+\cdot} \in A|\mathcal{F}_s) = P^{\eta_s}(A) \quad \text{for all } A \in \mathcal{F}_\infty.$$

If you are saying to yourself "the filtration needs to be made right-continuous by setting $\mathcal{F}_s^+ = \cap_{t>s}\mathcal{F}_t$" you probably don't need to read this introduction. If you don't know what this means don't worry, we will not make use of any sophisticatd results about stochastic processes.

Let $C(\mathbf{X})$ denote the continuous functions on X with the norm

$$\|f\| = \sup_{\eta \in X} |f(\eta)|.$$

When $\mathbf{X} = F^S$, the sequence f_n converges to f in $C(\mathbf{X})$ if for each $x \in S$, $f_n(x) \to f(x)$. Given a Markov process η_t we can define a **semigroup** by

$$S(t)f(\eta) = E^\eta f(\eta_t) \qquad \text{for } f \in C(\mathbf{X}).$$

The name comes from the fact that the Markov property implies $S(s + t) = S(s)S(t)$. We say that η_t is a **Feller process** if $f \in C(\mathbf{X})$ implies $S(t)f \in C(\mathbf{X})$. Here and throughout this section we are implicitly assuming that $\mathbf{X}$ is compact.

If μ is a probability measure then $\mu S(t)$ is the distribution of η_t when η_0 has distribution μ. μ is said to be a **stationary distribution** if $\mu S(t) = \mu$. Let $\mathcal{I}$ be the class of stationary distribution. The letter chosen makes more sense if you realize that these objects are also called **invariant measures**. The next result, which is part of Liggett's Proposition 1.8 is useful for constructing stationary distribution.

Ceslim **Theorem 1.1.1.** *(a) If* $\nu = \lim_{t\to\infty} \mu S(t)$ *then* $\nu \in \mathcal{I}$.
(b) If $\nu = \lim_{n\to\infty}(1/T_n)\int_0^{T_n} \mu S(t)\, dt$ *for some* $T_n \to \infty$ *then* $\nu \in \mathcal{I}$
(c) If $\mathbf{X}$ *is compact then* $\mathcal{I} \neq \emptyset$.

The averages in (b) are called **Cesaro averages**. Part (b) implies (c). Since $\mathbf{X}$ is compact the sequence of measures $\nu_n = (1/n)\int_0^n \mu S(t)\, dt$ is tight, so extracting a convergent subsequence gives the desired stationary distribution.

Generators

In most cases the the dynamics of the process will be specified by giving their **flip rates**, $c(x,\eta)$ which in the case of F^S with $|F| = 2$ is the rate that the state of x flips from $\eta(x)$ to $1-\eta(x)$ when the configuration is η. In the systems we consider $S = \mathbb{Z}^d$, the flip rates are **translation invariant and have finite range**, i.e, there is a finite set of neighbors $\mathcal{N}$ of 0 so that $c(x,\eta)$ can be determined from the values of $\eta(y)$ with $y \in x + \mathcal{N}$.

If S is finite the flip rates give rise to a **Q-matrix** that has

$$\frac{d}{dt} E_x f(Y_t) \bigg|_{t=0} = \sum_x Q(x,y)(f(y) - f(x))$$

When $\mathbf{X} = \{0,1\}^S$ with S is countable, we can define a **(pre)generator** on the collection of functions $\mathcal{D}(\Omega)$ that depend on finitely many coordinates by

$$\Omega f(\eta) = \sum_x c(x,\eta) f(\eta^x) - f(\eta)$$

where η^x is η with the value at x flipped. The **generator** L of the semigroup i$S(t)$ s defined by

$$\left\| \frac{S(t)f - f}{t} - Lf \right\| \to 0$$

with $\mathcal{D}(L)$, the domain of L, being the f for which the limit exists in the norm.

Much of Section 2 in Chapter 1 of Liggett (1985) is devoted to explaining when a Markov (pre)generator has a unique extension that is the generator of a Markov process. This result called the **Hille-Yosida theorem** is Theorem 2.9 in Chapter 1 of Liggett (1985). This analytic machinery is needed when one has processes with infinite range. It has the benefit of giving a simple condition that guarantee **ergocidity** of the Markov process, i.e., there is a unique stationary distribution that is the limit from any initial state.

We will take the much simpler approach of constructing our processes from an infinite family of Poisson processes. In Section 1.2 we use a construction of Harris (1972) that applies to any process that is translation invariant and has finitie range. In Section 1.3 we introduce a **graphical representation** that is less general but that has the advantage that when it applies then we can define a **dual process**.

1.2 Harris construction for finite range models

c:Harriscon

The first order of business is to construct the processes we will study. The state of an interacting particle system at time t a map $\xi : \mathbb{Z}^d \to F$ where F is a finite set of states. The dynamics are defined by **flip rates** $c_{ij}(x, \xi)$ which give the rate x changes from state i to state $j \neq i$ when the configuration is ξ. By definition the flip rate $c_{ij}(x, \xi) = 0$ if $\xi(x) \neq i$.

We assume the flip rates are finite range and translation invariant, so we can construct our process using a simple constrction due to Harris (1972), which will be useful when we develop the theory. By scaling time we can suppose that for all transitions i, j the filp rates $c_{ij}(x, \xi) \leq 1$ for all sites x and configurations ξ. In this case, we only need two things to construct the process

- Independent rate 1 Poisson processes $T_n^{x,i,j}$, $n \geq 1$, for $x \in \mathbb{Z}^d$ and $i, j \in F$.
- Independent $U_n^{x,i,j}$ uniform on $(0, 1)$, $n \geq 1$, for $x \in \mathbb{Z}^d$ and $i, j \in F$.

We will not use the words graphical representation here since they are reserved for the construction in the next section.

The process is constructed by a simple algorithm: If $T_n^{x,i,j} = t$ and $\xi_{t-}(x) = i$ (where the left limit indicates we are looking at the limit of the state as $s \uparrow t$), then $\xi_t(x) = j$ if $U_n^{x,j} < c_{ij}(x, \xi_{t-})$, otherwise $\xi_t(x) = i$ There is a small technical problem that must be overcome here: Since there are infinitely many Poisson processes there is no first arrival, and we need to show.

Theorem 1.2.1. *Our recipe specifies a unique process for any initial condition.*

Proof. We have assumed that our interations are finite range and translation invariant so there is a finite set $\mathcal{N}$ called the **neighbors of 0**, so that the flip rates $c_{ij}(0, \xi)$ can be determined if we know the values of $\xi(z)$ for $z \in \mathcal{N}$. Let $\epsilon > 0$ be small. If $T_1^x < \epsilon$ then draw unoriented edges $(x, x + z)$ for all $x \in \mathbb{Z}^d$, $z \in \mathcal{N}$ and call them **open**. Let $\mathcal{C}_x$ be the set of all points that can be reached from x by a path of open edges. If $\epsilon|\mathcal{N}| < 1$ then a simple branching process argument implies that with probability one all the $\mathcal{C}_x$ are finite. Since x only interacts directly or indirectly with sites in $\mathcal{C}_x$ and there is a first arrival among the Poisson processes associated with the $y \in \mathcal{C}_x$, by considering the changes associated with the Poisson arrivals in T_y, $y \in \mathcal{C}_x$ then we can compute the state of all sites x at time ϵ. Repeating the construction we can construct the process up to times $2\epsilon, 3\epsilon, \ldots$ □

Computation process

We will now describe a way to compute the state of site x at time t. The idea traces back to Section 2 of Durrett and Neuhauser (1994). Let $\mathcal{I}_s^{x,t}$ be set of sites at time $t - s$ that are needed to determine the state of x at time t. This is called the **influence set**. Initially $\mathcal{I}_s^{x,t} = \{x\}$ and we set $S_0 = 0$

For $m \geq 1$ let S_m be the smallest value of s larger than S_{m-1} so that there is an arrival at time $t - s$ in one of the processes $T_n^{y,i,j}$ for $y \in \mathcal{I}^{x,t}(S_{m-1})$. Let X_m be the site at which the arrival occurs, and i_m, j_m be the values of $i, j$and $V_m = U_m(X_m, i_m, j_m)$ Here an in what follows when the subscripts get too complicated we bring them up to the line and enclose them in parentheses. To keep track of what happened draw oriented edges from (X_m+y, S_m) to $(X_m, S_m i)$ for $y \in \mathcal{N}$, and we add the points $X_m + y$ to the influence set at time S_m.

When we can no longer find an S_m then the construction terminates and we let $K = m-1$ be the last be the index of the last time successuflly defined. We use the initial consition to assign values to the sites in $\mathcal{I}(S_K)$. We now work up the influence set using the values of X_k, i_k, j_k, U_k, $k < K$ an the flip rates to determine the state at time $S_{K-1}, \dots, S_0 = t$,

Adding stirring

In Chapter 4 we will add stirring in which the values at x and y are exchanged at rate $\lambda_{x,y}$. To incorportate this we add rate $\lambda_{x,y}$ poisson processes $T_n^{x,y}$ and at time $T_n^{x,y}$ draw an arrow from x to y and another arrow from y to x. When we work backward in time in the computation a lineage that hits x moves to y and a lineage that hits y moves to x.

1.3 Graphical representation, additive processes

ec:graphrep

Harris (1976), inspired by the existence of daul processes for the contact process and voter model, introduced a **graphical representation** for constructing interacting particle system. Like his construction described in the previous section it defines copies ξ_t^A of the process starting from each initial state A but now they have the **additivity property**

$$\xi_t^{A\cup B} = \xi_t^A \cup \xi_t^B \tag{1.3.1}$$ gradd

and, what is more important, it is possible to definite a **dual process** ζ_t^B so that

$$P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \zeta_t^B \neq \emptyset) \tag{1.3.2}$$ grdual

In all our processes the states $F = \{0, 1\}$. Sites in state 0 are vacant and in state 1 are occupied by particles. We begin by describing the construction for the

1.3.1. Basic contact process

Continuing to use the notation for flip rates used in the Harris construction, the rate of flips from 1 to 0, $c_{10}(x,\xi) \equiv 1$. In words particles die at rate 1. Let $\mathcal{N}$ be the set of neighbors, for example $\{x \in \mathbb{Z}^d : |z| = 1\}$ and let

$$c_{01}(x,\xi) = \lambda|\{z \in \mathcal{N} : \xi(x+z) = 1\}|$$

That is, the birth rate at x is proprtonal to the number of occupied neighbors. Because of this we can reformulate the dynamics as follows: each $y \in x + \mathcal{N}$ gives birth at rate λ to a particle sent to x. If x is vacant it becomes occupied, otherwise no change occurs.

For the basic contact process, we can reformulate the generic Poisson processes from Harris' construction as follows

(i) For each site there is an independent rate 1 Poisson process T_n^x, $n \geq 1$. At the arrivals of this process if there is a particle at x, it will die. To facilitate later definitions we write a dot ($\bullet$) at x at the times T_n^x. In LaTeX terms the bullet kills any particle at x

(ii) For each $z \in \mathcal{N}$ we have a rate λ Poisson process $T_n^{x,z}$. At these times we draw an arrow from $x+z$ to x to indicate that if $x+z$ is occupied there will be a birth from $x+z$ to x.

This structure with its dots and arrows is made to resemble oriented percolation. Fluid flows up through the graphical representation, crossing arrows in the direction of their orientation, but not through dots which are blockades. In the orignal notation δs were written next to dots to remind us that they killed particles but we simply write dots. Figure 1.1 gives a picture of the construction.

Let ξ_t^A denote the contact process starting from A occupied at time 0. A point $y \in \xi_t^A$ if for some $x \in A$ there is path from $(x,0)$ to (y,t) that goes up the graphical representation without passing through dots and crosses edges in the direction of their orientation. An

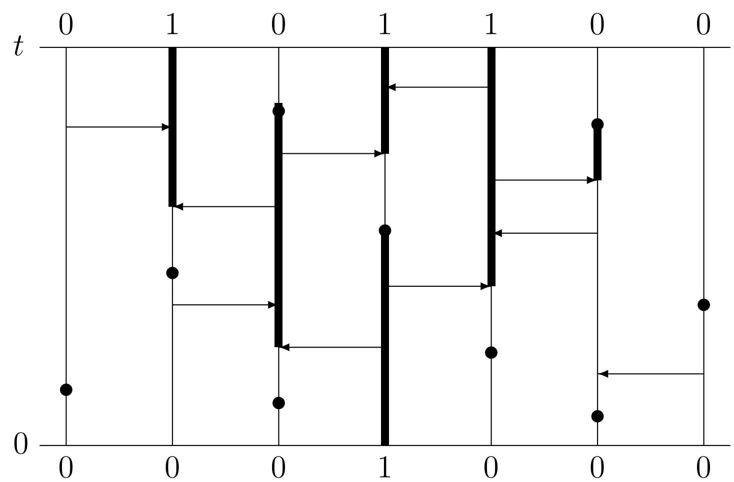


Figure 1.1: Graphical representation for the contact process. We think of fluid flowing up the structure and across arrows in the direction of the orientation, but being stopped by dots.Thick lines indicate occupied sites.

immediate conseqweunce of the the definition is that we have the additivity property in (1.3.1). Taking $C = A$ and $D = A \cup B$in that formula it follows that

$$\text{if } C \subset D \text{ then } \xi_t^C \subset \xi_t^D. \tag{1.3.3}$$ grmono

When a set-valued Markov process has this property, i.e., given initial conditions $C \subset D$ it is possible to define the two processes on the same space so that $\xi_t^C \subset \xi_t^D$ for all t it is called **attractive**.

Duality

An important consequence of the graphical representation is that it allows us to define for each x a **dual process** $\zeta_s^{x,t}$, $s \leq t$, that works backwards in time to answer the question "Is the site x occupied at time t?" The dual process can be constructed by a variant of the rule used for the process going forward in time: $y \in \zeta_s^{x,t}$ if there is a path from (x, t) to $(y, t - s)$ that goes down the graphical representation without passing through dots and crossses edges in the direction OPPOSITE their orientation. A little thought reveals that if the neighborhood is symmetric, i.e., $\mathcal{N} = -\mathcal{N}$ then the dual of the contact process has the same distribution as the original.

We extend the definition of the dual starting from a point x to starting from an initial set B by setting

$$\zeta_s^{B,t} = \cup_{x \in B}\, \zeta_s^{x,t}.$$

A little thought shows that

$$\{\xi_t^A \cap B \neq \emptyset\} = \{A \cap \zeta_t^{B,t} \neq \emptyset\}. \qquad (1.3.4)$$ cdualeq

In words there is a path up from some $(x, 0)$ with $x \in A$ to some (y, t) with $y \in B$ if and only if there is a path down from some (y, t) with $y \in B$ to some $(x, 0)$ with $x \in A$.

The almost sure equality in (1.3.6) is convenient for establishing the equation but it is useful to rewrite the equality without the superscript t. To do this we note that if $t < t'$ then the joint distribution of the $\zeta_s^{x,t}$, $s \leq t$ with $x \in \mathbb{Z}^d$ is the same as that of $\zeta_s^{x,t'}$, with $x \in \mathbb{Z}^d$ when $s \leq t$, so using the Kolmogorov extension theorem there is a processes ζ_s^x whose joint distributions agree with $\zeta_s^{x,t}$ on $s \leq t$.

$$P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \zeta_t^B \neq \emptyset). \qquad (1.3.5)$$ setdual

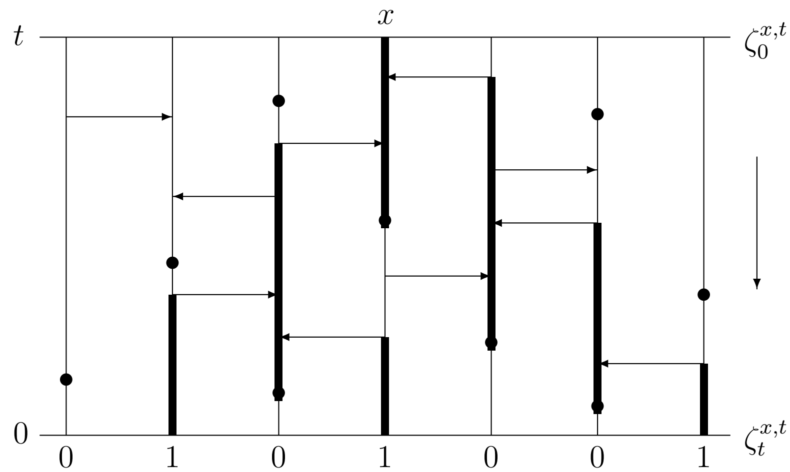


Figure 1.2: Dual of the contact process.

Remark. At the beginning of this chapter we said that the graphical representation is, in essence, an oriented percolation process on $\mathbb{Z}^d \times (-\infty, \infty)$. To explain this remark let $\mathcal{L}_\epsilon = \{(x, n\epsilon) : x, n \in \mathbb{Z}\}$ and introduce edges from $(x, n\epsilon) \to (x, (n+1)\epsilon)$ that are open with probability $1 - \epsilon$ and edges from $(x, n\epsilon) \to (x+1, (n+1\epsilon)$ and from $(x, n\epsilon) \to (x-1, (n+1\epsilon)$ that are open with probability $\lambda\epsilon$. In the limit as $\epsilon \to 0$ the closed vertical bonds become a Poisson process of holes at rate 1, while the open bonds that connect x to $x + 1$ become a Poisson process of arrows at rate λ.

1.3.2. Contact processes with nonlinear birth rates

In this class of processes we keep the simple death rate: particles die at rate 1, and we generalize the birth rate. The process can be constructed by adapting the previous definition as follows. The first part of the definition remains the same.

(i) For each site there is an independent rate 1 Poisson process T_n^x, $n \geq 1$. We write a dot at x at the times T_n^x to indicate that if there is a particle at x, it will die.

(ii) For each $A \subset \mathcal{N}$ we have a rate λ_A Poisson process $T_n^{x,A}$. At these times we draw an arrow from $x+z$ to x for each $z \in A$ to indicate that if any of the $x+z$, $z \in A$ is occupied there will be a birth at x.

Threshold 1 contact process. This is perhaps the simplest example of nonlinear birth rates. For each x, we only have one birth Poisson processes for $A = \mathcal{N}$. The result is that at an arrival there is a birth at x if at least one neighbor is occupied. This gives rise to a dual process that gives births onto all of the neighbors.

One dimensional nearest neighbor case. To see what birth rates can be produced by a graphical representation define the λ_A by

$$\lambda_{\{-1\}} = \lambda_{\{1\}} = a \qquad \lambda_{\{-1,1\}} = b$$

Let the birth rate by $c_{01}(0,\eta) = \lambda_i$ if $|\{z \in \mathcal{N} : \xi(z) = 1\}| = i$. In plain English the birth rate at x is λ_i if i neighbors are occupied. Since $\lambda_1 = a + b$ and $\lambda_2 = 2a + b$ itis easy to see that there is a choice of a and b to construct these rates if and only if

$$\lambda_1 \leq \lambda_2 \leq 2\lambda_1$$

since we must have $a = \lambda_2 - \lambda_1$ and $b = 2\lambda_1 - \lambda_2$. No other pairs of rates (λ_1, λ_2) can be constructed from our graphical representation.

Four neighbors. Consider now the case in which $|\mathcal{N}| = 4$. The dimension of the space is irrelevant but perhaps the most naural examples are $d = 1$, $\mathcal{N} = \{-2,-1,1,2\}$ or $d = 2$ and the four nearest neighbors $\mathcal{N} = \{(1,0),(0,1),(-1,0),(0,-1)\}$. To build the process we suppose that each gadget with arrows from k neighbors has rate a_k, and make the following table to show the birth rate at x due to the sets of a given size

$\|A\|$	number of sets A	1	2	3	4
1	$C_{4,1} = 4$	a_1	$2a_1$	$3a_1$	$4a_1$
2	$C_{4,2} = 6$	$3a_2$	$5a_2$	$6a_2$	$6a_2$
3	$C_{4,3} = 4$	$3a_3$	$4a_3$	$4a_3$	$4a_3$
4	$C_{4,4} = 1$	a_4	a_4	a_4	a_4

The numbers 1, 2, 3, and 4 on th first row indicate the number of occupied neighbors. The entries below them are birth rates in this case due to the gadgets with k arrows pointing to x. To explain the row for sets of size 2, note that

> if there is one occupied neighbor there are 3 sets that contain it and 3 that do not,
> if there are two occupied neighbors then only 1 set does not contain at least one,
> if three or more neighbors are occupied it is impossible to avoid all of them.

By adding up the entries in the column labeled i we get λ_i. To compute the coefficients a_i from the λ we note that

$$\begin{aligned}
a_1 &= \lambda_4 - \lambda_3 \\
a_2 &= -\lambda_4 + 2\lambda_3 - \lambda_2 \\
a_3 &= \lambda_4 - 3\lambda_3 + 3\lambda_2 - \lambda_1 \\
a_4 &= -\lambda_4 + 4\lambda_3 - 6\lambda_2 + 4\lambda_1 - \lambda_0
\end{aligned}$$

$\lambda_0 = 0$ but is included to more clearly show the pattern. Of course, we need the $a_k \geq 0$ to be able to construct the process. To see the pattern in the equations note that $f(n) - f(n-1)$ is the slope, $f(n) - 2f(n-1) - f(n-2)$ is a discrete second derivativee, $f(n) - 3f(n-1) + 3f(n-2) - f(n-3)$ is a discrete third derivative, etc. Thus the derivatives have the alteranting signs that characterize a completely monotone function.

In Section 6 of Harris (1976) it was show that when $|\mathcal{N}| = n$ the $a_k \geq 0$ if for all $1 \leq k \leq n$ we have

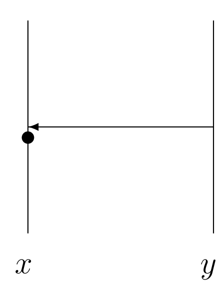
$$\sum_{j=0}^{k} (-1)^{1+j} \binom{n}{j} \lambda_{n-k+j} \geq 0$$

Taking $n = 4$ gives our conditions here. When $n = 2$ we get our first example.

1.3.3. Voter models

The voter model was introduced independently by Clifford and Sudbury (1973) and Holley and Liggett (1975) on the d-dimensional integer lattice. It is a very simple model for the competition of two opinons and has been investigated in great detail, see part II of Liggett's (1999) book for a survey. We skip the verbal description and go right to the construction. For each site x there is an independent rate 1 Poisson processes, T_n^x, $n \geq 1$. At the times T_n^x, $n \geq 1$ the individual at x picks a neighbor $Z_n^x \in \mathcal{N}$ at random, and at time $t = T_n^x$ we set $\xi_t(x) = \xi_t(x + Z_n^x)$.

We can construct the voter model from a graphical representation if we allow dots and arows to occur at the same time.. At time T_n^x, $n \geq 1$ we put a $\bullet$ at x. We let Z_n^x be chosen at random from $\mathcal{N}$ and draw an arrow from $x + Z_n^x \to x$, the arrow arriving just after the $\bullet$ has killed any particle at x. Writing y as shorthand for $x + Z_n^x$.



A little thought reveals that the combination of these two events causes the following changes to occur.

before x	y	after x	y
0	0	0	0
0	1	1	1
1	0	0	0
1	1	1	1

To check the table, note that the • kills any particle at x while the arrow gives birth from y to x if there is a particle at y. On the last line the • kills the particle at x, but then the arrow restores it. No change occurs at y. The result of these two mechanisms is a voter event: x imitates the value at y.

This construction allows us to define a **dual process** by working down the graphical representation and crossing arrows in the direction opposite their orientation. The combination of the two events causes the following changes in the dual process.

before x	y	after x	y
0	0	0	0
0	1	0	1
1	0	0	1
1	1	0	1

To check the table ,note that if x is in state 0 nothing happens, while if x is in state 1 then the arrow from y to x which is crossed in the direction opposite its orientation, gives birth to a particle at y and then the • kills the 1 at x. The result is a **coalescing random walk** step: the particle at x jumps to y, and if y is occupied the two particles coalesce.

At this point we have defined dual of the voter model starting from x at time t, $\zeta_s^{x,t}$ and found it to be a coalescing random walk. As with the contact process

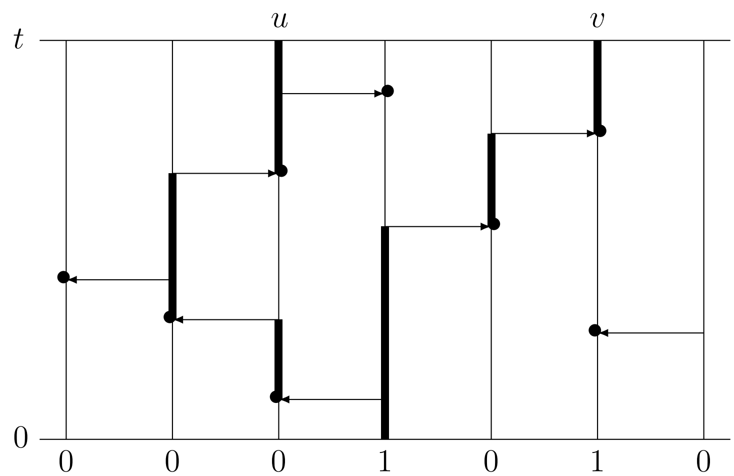
(i) we extend the definition of the dual starting from a point x to starting from an initial set B by setting $\zeta_s^{B,t} = \cup_{x\in B}\, \zeta_s^{x,t}$.

(ii) From the definition of the dual it follows that

$$\{\xi_t^A \cap B \neq \emptyset\} = \{A \cap \zeta_t^{B,t} \neq \emptyset\}. \tag{1.3.6}$$ cdualeq

(iii) Using Kolmogorov's extension theorem we can define ζ_s^x whose joint distributions agree with $\zeta_s^{x,t}$ on $s \leq t$ nd we have the folllowing equality

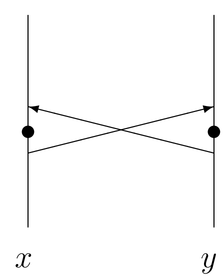
$$P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \zeta_t^B \neq \emptyset). \tag{1.3.7}$$ setdual



g:voterdual Figure 1.3: A realization of the dual process for the voter model. Particles start at u and v performing random walks that jump across an arrow when they encounter its head. When tow particles hit they coalesce to 1. When we get down to time 0 we see that the states of u and v are the same and are equal to 1.

1.3.4. Stirring

Stirring, which is the exchange of the values at two sites x and y, is a double voter event. x takes on the old value at y and y takes on the old value at x



Here the arrows should start just before one dot and end just after the other one but this is the best we can do in the LaTeX picture environment. A little thought reveals that the combination of these two events causes the following changes to occur.

before x	y	after x	y
0	0	0	0
0	1	1	0
1	0	0	1
1	1	1	1

If we view the 1's are unlabled particles then this implements the **simple exclusion process.** Particles perform independent randomd walks subject to the exclusion rule. If the particle tries to jump to an occupied site then the jump is suppressed. This is the third process considered in Liggett's 1999 book. If the particles are labelled in some way, e.g., they are diffferent colors then the recipe fails since in the last case the particles exchange locations.

As in the voter model the construction allows us to define a dual process by working down the graphical representation. It is easy to see that the stirring process, like the contact process is self-dual.

1.3.5. Biased voter model

This is a very simple example: we take a voter model and add contact process births.

(i) For each $x \in \mathbb{Z}^d$ and nearest neighbor y we have a rate 1 Poissso process $T_n^{(x,y)}$, $n \geq 1$. At arrival times we write a $\bullet$ at x and draw an arrow from y to x.

(ii) For each $x \in \mathbb{Z}^d$ and nearest neighbor y we have a rate $\lambda - 1$ Poissso process $S_n^{(x,y)}$, $n \geq 1$. At arrival times we draw an arrow from y to x.

The combined effect of these two arrow is that on each discordant nearest neighbor edge, i.e., one connecting a 0 to a 1, the 0 converts the 1 to a 0 at rate 1 while the 1 converts the 0 to a 1 at rate $\lambda > 1$.

Suppose we start the process with a single 1 at the origin and call it ξ_t^0. If at time t the number of occupied sites $|\xi_t^0| = n$ and the number of discordant edges is m then the $|\xi_t^0|$

$$\begin{array}{cc} \text{jumps} & \text{at rate} \\ n \to n+1 & \lambda m \\ n \to n-1 & m \end{array}$$

This implies that the number of 1's a time change of random walk that jumps up with probability $\lambda/(\lambda+1)$ and down with probability $1/(\lambda+1)$. If we let $r = 1/\lambda$ then

$$r^n = r^{n-1} \cdot \frac{1}{1+\lambda} + r^n \cdot \frac{\lambda}{1+\lambda}$$

so $r^{|\xi_t^0}]$ is a martingale and if $\lambda > 1$

$$P(|\xi_t^0| = 0 \text{ for some } t) = 1/\lambda$$

1.3.6. Nonlinear voter models

We have seen that the existence of a dual places strong restrictions on the birth rates of a nonlinear contact process. The restrictions are even more severe on symmetric nonlinear voter models.

Claim. *Using the graphical representation one cannot construct a voter model in which the flip rates depend only on the number of neighbors with the opposite opinion n_x and are nonlinear.*

Proof. For simplicity, we only prove the result when the neighborhood has size 4. The only gadgets than can be used in the graphical representation are combination of arrows and δ's. To begin, we will consider the set of processes that can be constructed by only using gadgets that have a δ at x and a number of arrows that point to x from its neighbors. We call these objects arrow-δs. Since the flip rates only depend on the number of sites, all arrow-δs with k arrows must have the same rate, a_k.

- When there is a 1 at x the δ will cause the 1 to flip to a 0. However, the site will only stay a 0 if *all* neighbors connected to x by arrows are in state 0.
- When there is a 0 at x then the δ does nothing, and the site will flip to 1 if there is *at least one* neighbor in state 1 connected to x by an arrow.

The number of k-arrow gadgets is $\binom{4}{k}$ so the flip rates are as follows

n_x	rate $1 \to 0$	rate $0 \to 1$
0	0	0
1	a_1	$a_1 + 3a_2 + 3a_3 + a_4$
2	$2a_1 + a_2$	$2a_1 + 5a_2 + 4a_3 + a_4$
3	$3a_1 + 3a_2 + a_3$	$3a_1 + 6a_2 + 4a_3 + a_4$
4	$4a_1 + 6a_2 + 4a_3 + a_4$	$4a_1 + 6a_2 + 4a_3 + a_4$

If we add δ's with no arrows then they will flip 1s even when all their neighbors are 1. If a_2, a_3, or a_4 is positive the rate of flipping $1 \to 0$ is $<$ the rate of flipping $0 \to 1$. when $n_x = 1, 2, 3$. Adding arrows with no δs will only further increase the rates of flips $0 \to 1$. □

1.3.7. Function-based duality

Our treatment of duality would not be complete unless we explained that for many people duality is a purely analytic concept. Markov processes X_t and Y_t are said to be **in duality with respect to** H if

$$EH(X_t, Y_0) = EH(X_0, Y_t). \tag{1.3.8}$$ Hdual

If the two processes are set valued and $H_a(\xi, \zeta) = 1$ when $\xi \cap \zeta \neq \emptyset$ (and 0 otherwise) this gives **additive duality** we have discussed in this section

$$P(\xi_t^A \cap B \neq \emptyset) = P(A \cap \zeta_t^B \neq \emptyset).$$

If the two processes are set valued and $H_c(\xi, \zeta) = 1$ when $|\xi \cap \zeta|$ is odd (and 0 otherwise) this gives **cancellative duality**

$$P(|\xi_t^A \cap B| \text{ is odd}) = P(|A \cap \zeta_t^B| \text{ is odd}).$$

This holds for a number of models, see Griffeath's (1978) book: annihilating random walks, some nonlinear voter models included the Neuhauser-Pacala model, and some instances of the Ising model. It is less useful than the additive case since it is harder to check $|\xi_t^A \cap B|$ is odd than it is to check $|\xi_t^A \cap B| > 0$, since in the latter case a few extra particles doesn't hurt.

Moment duality refers to the case: $H_m(y,n) = y^n$ where $y \in [0,1]$ and n is a positive integer. The most important case for us occurs in Chapter 2 when Y_t is a Wright-Fisher diffusion and and D_t is the pure death process known as Kingman's coalescent.

$$E_\theta(Y_t^m) = E_m(\theta^{D_t}) \tag{1.3.9}$$ WFKcdual

Proof. Proposition 1.2 in Jansen and Kurt (2014) implies that (1.3.8) holds if

$$\mathcal{L}_y^Y H(y,n) = \mathcal{L}_n^D H(y,n), \tag{1.3.10}$$ suffHdual

where the generator subscripts indicate the variables they act on (with the other variable held constant).

Wright-Fisher diffusion generator is $\mathcal{L}^Y f(y) = (1/2)\gamma y(1-y)f''(y)$

Kingman's coalescent generator is $\mathcal{L}^D f(n) = \gamma\binom{n}{2}(f(n-1) - f(n)$.

To check the Wright-Fisher diffusion-Kingman coalescent duality now we note that if $n \geq 2$

$$\begin{aligned}
\mathcal{L}_y^Y H(y,n) &= \frac{1}{2}\gamma n(n-1)y^{n-2}y(1-y) \\
&= \gamma\frac{n(n-1)}{2}(y^{n-1} - y^n) = \mathcal{L}_n^N H(y,n).
\end{aligned}$$

If $n = 1$ then all of these quantities are 0. □

1.4 Attractive processes, correlation inequalities

Any process construed on the graphical representation has the property :

if $A \subset B$ then $\xi_t^A \subset \xi_t^B$.

We say that a particle system is **attractive** if given initial states $A \subset B$ then the two processes can be constructed on the same space so that $\xi_t^A \subset \xi_t^B$. According to Theorem 2.2 on page 134 of Liggett (1985) the following is a necessary and sufficient condition to be attractive

$$\eta \leq \zeta \quad \text{implies} \quad \begin{cases} c_{01}(x,\eta) \leq c_{01}(x,\zeta) & \text{if } \eta(x) = \zeta(x) = 0 \\ c_{10}(x,\eta) \leq c_{10}(x,\zeta) & \text{if } \eta(x) = \zeta(x) = 1 \end{cases}$$

The restriction to $\eta(x) = \zeta(x)$ is there because the inequality does not have to hold when $\eta(x) = 0$ and $\zeta(x) = 1$. When the two process differ at a stie we let them make independent jumps.

An important consequence of a process being attractive is

upinvm **Theorem 1.4.1.** *If we let ξ_t^1 be the system starting from all sites occupied then ξ_t^1 converges in distribution to a limit ξ_∞^1, which is a stationary distribution.*

The fact that the limit is stationary follows from Theorem 1.1.1. Due to monotonicity property in (1.3.3), ξ_∞^1 is the largest possible stationary distribution. If one starts with no particles, a process that we will denote by $\xi_t^\emptyset$ then the limit $\xi_\infty^\emptyset$ is the smallest possible stationary distribution. (We do not write ξ_∞^0 since that is the process starting from a particle at 0.) If $\xi_\infty^1 = \xi_\infty^\emptyset$ then the stationary distribution is unique. Many of the processes we deal with have all 0's as a absorbing state, so this stationary distribution is trivial, $\xi_\infty^\emptyset = \delta_0$ the point mass on the all 0's configuration. In this case, if $\xi_\infty^1 = \delta_\emptyset$ then there are no nontrivial stationary distributions.

Using the self-duality of the contact process in (1.3.7) with $A = \mathbb{Z}^d$ and $B = \{x\}$ implies that

$$P(x \in \xi_t^1) = P(\xi_t^x \neq \emptyset)$$

Since $\emptyset$ is an absorbing state for the contact process, the right-hand side decreases to a limit as $t \to \infty$. From this we see that the contact process has a nontrivial stationary distribution if and only if it has positive probability of surviving starting from a single point. Furthermore, the density of 1's in equilibrium is equal to the probability of nonextinction.

Positive correlations

In the theory of percolation the following result first proved by Harris (1960) is very useful. See e.g., Grimmett (1999) Section 2.2. Suppose we have a collection of i.i.d. random variables e_i that take only the values 0 or 1, for example the random variables that indicate whether bonds in percolation are open (1) or closed (0). If $P(e_i = 1) = p$ we use P_p to denote the **product measure with density** p and E_p to denote the corresponding expected value. Introduce a partial order by declaring that $e \leq e'$ if $e_i \leq e'_i$ for all i

Harris60 **Theorem 1.4.2.** *Suppose that X and Y are increasing functions of the e_i in the partial order just defined, and that $E_p X^2, E_p Y^2 < \infty$. Then*

$$E_p(XY) \geq E_p(X) \cdot E_p(Y)$$

Consequently if A and B are increasing events

$$P(A \cap B) \geq P(A)P(B)$$

Harris' (1977) theorem

Writing things in a small perturbation of his original notation, let E be a finite set with a partial ordering $\leq$. Call f **increasing** if $x < y$ implies $f(x) \leq f(y)$ and let C_{inc} be the set of all increasing functions. We say that μ has **positive correlations** if $\mu(fg) \geq \mu(f)\mu(g)$ for all $f, g \in C_{inc}$. Let $\mathcal{M}_{pc}$ be the set of μ with positive correlations.

Let X_t, $t \geq 0$ be a Markov process with step-function paths (today we would call them cádlág, French for rught continuous with left limits) in the finite state space E, a transition probability $p(t, x, y)$, and semigroup $T_t f(x) = \sum_y p(t, x, y) f(y)$. Call X_t or T_t monotone if $T_t C_{inc} \subset C_{inc}$. Let $U_t \mu(y) = \sum_x \mu(x) p(t, x, y)$.

Harrispc **Theorem 1.4.3.** *Let X_t be an attractive process in a finite partially ordered state space E. In order that $U_t \mathcal{M}_{pc} \subset \mathcal{M}_{pc}$ for each $t > 0$ it is necessary and sufficient that each jump is up or down in the partial order on E.*

In the contact process and voter model only one site changes at a time so each jump is up or down. However, particle jumps in the stirring process are neither up nor down. The next result is an immediate conseqeunce of the definitions.

corpc **Theorem 1.4.4.** *Let f and g be increasing functions on E. If X_0 has positive correlations then $f(X(t))$ and $g(X(t))$ are positively correlated.*

Note that by Theorem 1.4.2 a product measure has positive correlations, so the upper invariant measure for the contact process and equilibria ν_p for the voter model in $d \geq 3$ have positive correlations.

Negative correlations

Positive correlations leads to lower bounds on probabilities but in some situations we need upper bounds. In this situation the ineqaulity in Lemma 1.4.5 is useful. We say that A occurs on a collection of sites U if the fact that A occurs can be determined by looking at the coordinates in U. An important example is $x \sim y$, i.e., x is connected to y. A occurs on U if there is a path of sites in U connecting x to y. We say that A and B occur disjointly, and write $A \Box B$ if there are disjoint sets of sites U and V (which may depend upon the outcome) so that A occurs on U and B occurs on V.

BKineq **Lemma 1.4.5. van den Berg - Kesten inequality.** *If E_1 and E_2 are increasing events with respect to a percolation process then*

$$P(E_1 \Box E_2) \leq P(E_1)P(E_2)$$

Intuitively if E_2 has to occur disjointly from E_1 then $P(E_2|E_1) \leq P(E_2)$since the disjoint occurence requirement means there is less room for the event to happen.

extension to more general events - look at Grimmett.

Russo's inequality

The next inequality does not really fit under the heading of correlation inequalities but it is part of what we could call "useful results that were created to anlsyze percolation." A bond b is said to be **pivotal for** E, if E holds when the bond is occupied ($n_b = 1$) and not if the bond is vacant ($n_b = 0$). Note that the event b is pivotal is independnet of E.

Russo **Lemma 1.4.6. Russo's formula.** *If E is increasing then for each bond b with $p_b > 0$*

$$\frac{\partial P(E)}{\partial p_b} = P(b \text{ is pivotal for } E) = \frac{1}{1-p_b} P(n_b = 0, b \text{ is pivotal for } E)$$

The main use of this result is to show that certain events have probabilities that are a very steep function of p near the critical value.

1.5 Oriented percolation

We begin by describing **oriented bond percolation** in two dimensions with independent bonds. We will use the lattice

$$\mathcal{L} = \{(x,n) : x+n \text{ is even}, n \geq 0\}.$$

because it has the property that if two paths cross then they will intersect. Here x is space and n is time. We have oriented bonds $(x,n) \to (x+1,n+1)$ and $(x,n) \to (x-1,n+1)$ that are independently open with probability p, and closed otherwise. We define the cluster containing the origin to be

$$\mathcal{C}_0 = \{(y,n) : (0,0) \to (y,n)\},$$

where $(0,0) \to (y,n)$ is short for there is a path of open bonds from $(0,0)$ to (y,n). The path will have the form (x_i,i) where $x_0 = 0$, $x_n = y$ and $x_i = x_{i-1} \pm 1$ for $1 \leq i \leq n$. We think of introducing a fluid at at $(0,0)$ that can move up through open bonds, so we say that all the sites in $\mathcal{C}_0$ are wet.

We will also consider **oriented site percolation** in which it is the sites $z \in \mathcal{L}$ that are designated as open $\eta(z) = 1$ or closed $\eta(z) = 0$. In this case $(0,0) \to (y,n)$ holds if there is a path $(0,0)$ to (y,n). so that all the points (i,y_i) with $i > 0$ are open. The state of $(0,0)$ is set by the initial condition, so we do not need for the site to be open. This convention is nice because it guarantees $0 \in \mathcal{C}_0$.

Existence of a phase transition

Our first goal is to show that there is a $p < 1$ so that $P_p(|\mathcal{C}_0| = \infty) > 0$. When $|\mathcal{C}_0| = \infty$ we say that **percolation occurs**. By monotonicity this implies that there is a $p_c < 1$ so that percolation has positive probability for $p > p_c$ and has zero probability for $p < p_c$. To get a slightly better bound on p_c we replace $\mathcal{C}_0$ by

$$\mathcal{C}_K = \{(y,n) : (-k,0) \to (y,n) \text{ for some even integer } k \in [0,2K]\}.$$

See Figure 1.4. Note that there are $K+1$ initial wet sites in $\mathcal{C}_K$. Let D be the diamond with vertices $(-1,0)$, $(0,1)$, $(1,0)$ and $(0,-1)$. Define the **wet region**

$$W = \cup_{z \in \mathcal{C}_K} z + D,$$

and define the **contour** Γ to be the boundary of the unbounded component of $(\mathbb{R} \times (-1,\infty)) - W$, and orient Γ so that the segment from $(0,-1) \to (1,0)$ which is always present, is oriented in the direction indicated.

The picture in Figure 1.4 could be the wet region in bond or site percolation, but the probability of the picture is differentin the two cases. In order for the contour to form, sites 1 and 2 have to be closed to stop the progress for the fluid, while 3 and 4 do not have to be closed since paths are constrained to only move upward. The contour begins at $(-1,0)$ and

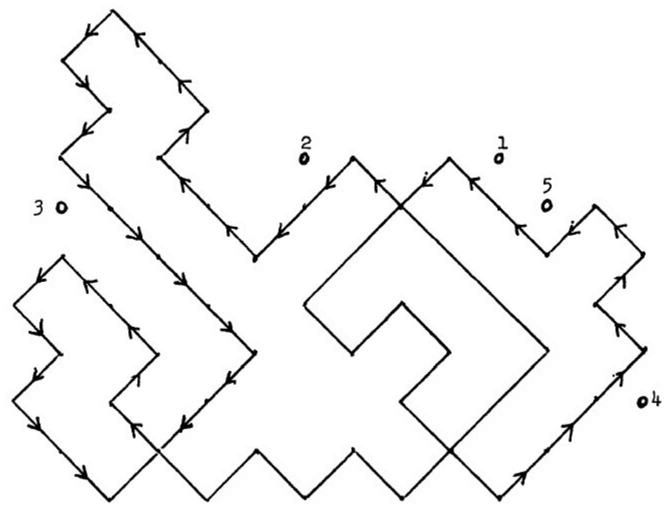


Figure 1.4: Percolation argument. $K = 4$, arrows indicate the contour $|Gamma$ fig:percarg

ends at $(-1, -2K)$. The segments associated with 1 and 2 move to the right one unit, while those associated with 3 and 4 move to the left one unit. If we let n_L and n_R be the number of paths of the two types then $n_L - n_R = 2K$ so if the total length of the path is $n = 2m$ then $n_L = m + K$ and $n_R = m - K$. From this we conclude that $n_L \geq n/2$.

In site percolation a closed site like 5 on the picture can be associated with two segments of the contour. so if the length is n then there are at least $n/4$ closed sites associated with the contour. In bond percolation there are two closed bonds that end at 5 so the number of closed bonds is $\geq n/2$.

The next detail is to count the number of contours. The first segment is always the same: $(-1, 0) \to (1, 0)$ After that the contour cannot go back in the direction it came from so there are at most 3^n contours. The shortest contour has length $2K + 4$ so

$$P(|\mathcal{C}_K| < \infty) \leq \sum_{n=2K+4}^{\infty} 3^n (1-p)^{n/4} = \frac{[3(1-p)^{1/4}]^{2K+4}}{1 - [3(1-p)^{1/4}]} \qquad (1.5.1)$$ CKest

If $3(1-p)^{1/4} < 1$ the sum is always finite and will be < 1 if we pick K large enough. This shows that for site percolation $p_c < 1 - 3^{-4}$. For bond percolation site 5 is associated with two closed bonds so we end up with $(1-p)^{n/2}$ and a bound of $p_c < 8/9$.

The bounds can be improved but that is a waste of time. You will never get close to the values that have been numerically computed to be 0.644701 for bond percolation and 0.705489 fior site percolation in what physicists call 1+1 dimensions. For us it is enough to know $p_c < 1$, which is fortunate since the bounds will get ridiculously bad when we allow dependence.

M-dependent site percolation

We say the the 0,1 valued random variables $\eta(m,n)$ are M-dependent with density at least $1-\gamma$ if whenever (m_i,n_i), $1 \le i \le k$ have $(|m_i - m_j| + |n_i - n_j|)/2 > M$ for all $i < j$ then

$$P(\eta(m_i,n_i) = 0 \text{ for } 1 \le i \le k) \le \gamma^k$$

Classical M-dependence would require that for the (m_i,n_i) considered above $\eta(m_i,n_i)$ would be independent, but in our definition we only consider the only probability we need to control. Recall that in the contour argument we need a set of boundary sites to be vacant in order for the contour to form.

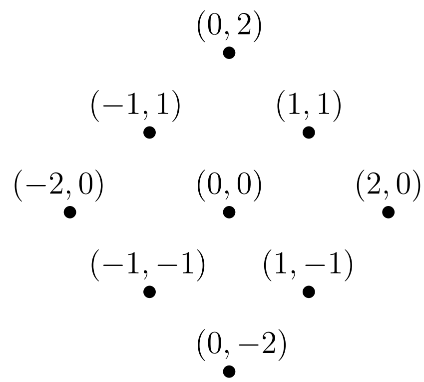


Figure 1.5: Sites within distance 1 of (0,0) on $\mathcal{L}$. There are $2M+1)^2$ within distance M,

We are now ready to state Theorems 4.1 and 4.2 from Durrett (1995), which concern M-dependent percolation. Unfortunately the first result has a typo in the reference just given with an undefined θ replacing γ in the conclusion

Block1 **Theorem 1.5.1.** *Let* $\gamma_M = 6^{-4(2M+1)^2}$. *If* $\gamma \le \gamma_M$ *then*

$$P(|\mathcal{C}_0| < \infty) \le 55\gamma^{1/(2M+1)^2} \le 1/20.$$

The proof is almost identical to the one in the independent case given above, but we have to reduce the set of closed sites adjacent to the contour even furthe so they all have distance $> M$.

For the next result we suppose that the initial condition W_0 considered above is independent and wet with probability $p > 0$, and we denote the percolation process by W_m^p.

Block2 **Theorem 1.5.2.** *If* $\gamma \le \gamma_M$ *then*

$$\liminf_{n\to\infty} P(0 \in W_{2n}^p) \ge 1 - 55\gamma^{1/(2M+1)^2} \ge 19/20.$$

In most cases, we will use this result with $p = 1$, where we know that the limit exists and defines the upper invariant measure.

When we use the block construction we encounter oriented percolation that is M-dependent with density close to 1. In most cases we can establish the results we need about the system using the contour method. However, Liggett, Schomann, and Stacey (1997) have found a more elegant solution by proving a result that allows us to return to the independent case. They consider 0,1 valued random that have the property that conditioned on what happens outside the neighborhood of a site, the probability the site is 1 is at least p. They show that if p is close to 1, these random variables dominate a product measure with density $p' = f(p)$ where $f(p) \to 1$ as $p \to 1$.

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